

Entrapment of a Network of Domain Walls

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We explore the idea of a network of defects to live inside a domain wall in models of three real scalar fields, engendering the $Z_2 \times Z_3$ symmetry. The field that governs the Z_2 symmetry generates a domain wall, and entraps the hexagonal network formed by the three-junctions of the model of two scalar fields that describes the remaining Z_3 symmetry. If the host domain wall bends to the spherical form, in the thin wall approximation there may appear non-topological structures hosting networks that accept diverse patterns. If Z_3 is also broken, the model may generate a buckyball containing sixty junctions, a fullerene-like structure. Applications to cosmology are outlined.

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Domain walls appear in diverse branches of physics, as for instance in systems of condensed matter that present ferromagnetic [1], ferroelectric [2] and other properties [3], and also in cosmology [4,5]. They arise in systems with at least two isolated degenerate minima, and in field theory they usually live in three spatial dimensions as bidimensional objects, seen as immersions into $(3,1)$ dimensions of static solutions of $(1,1)$ dimensional models that engender the Z_2 symmetry. The standard domain wall presents no internal structure, but there are models where they may entrap field configurations that engender non-trivial behavior. This idea follows as in Refs. [6–8], and in the more recent Refs. [9,10]. Other investigations include for instance supersymmetry [11], supergravity [12], and the recent applications to polymers [13].

Although domain walls may be dangerous [4,5] to cosmological applications, they have found their way into cosmology as for instance seeds for the formation of non-topological structures. This possibility appears in Refs. [14–17], where the discrete symmetry is changed to an approximate symmetry, or in Ref. [18], with the discrete symmetry biased so that domains of distinct but degenerate vacua spring unequally. The non-topological structures may be stable, but now stability requires the presence of conserved charges, of bosonic and/or fermionic origin.

Domain walls may also be of interest when they host non-trivial structures. We illustrate this point using the model introduced in the first work of Ref. [10], describing the pair of fields (ϕ, χ) via the superpotential $W(\phi, \chi) = -\phi + (1/3)\phi^3 + r\phi\chi^2$. Here r is a parameter, real and dimensionless, that couples the two fields. The system is described by a quartic potential, and we use natural units, working with dimensionless space and time variables, and fields. The equations of motion for field configurations $\phi = \phi(z)$ and $\chi = \chi(z)$ are solved by solutions of the first order differential equations $d\phi/dz = -1 + \phi^2 + r\chi^2$ and $d\chi/dz = 2r\phi\chi$. In this model, the sector connecting the minima $(\pm 1, 0)$ is a BPS sector, with energy density or tension $t = 4/3$. For

$r > 0$ this BPS sector admits two different types of static solutions: the one-field solutions $\phi_1(z) = \tanh(z)$ and $\chi_1 = 0$, and the two-field solutions $\phi_2(z) = -\tanh(2rz)$ and $\chi_2(z) = a(r)/\cosh(2rz)$, with $a^2(r) = 1/r - 2$, valid for $0 < r < 1/2$. We are working in $(3,1)$ space-time dimensions, so the one-field solution represents a standard domain wall, while the two-field solution appears as a domain wall having internal structure. As z varies in $(-\infty, \infty)$, in configuration space the vectors (ϕ_k, χ_k) , $k = 1, 2$, describe a straight line segment ($k = 1$) and an elliptic arc ($k = 2$), resembling light in the linearly and elliptically polarized cases, respectively. These solutions also appear in condensed matter, and there they are named Ising and Bloch walls, respectively [1–3]. They appear as solutions of the anisotropic XY system, a model used to describe ferromagnetic transition in magnetic systems. The Bloch walls can be seen as chiral interfaces, and may be used to describe more complex phenomena, as for instance in the applications where chirality is also broken [19].

In Ref. [20] the idea of nesting a network of defects inside a domain wall has been presented. This possibility may appear in models with three real scalar fields, engendering the $Z_2 \times Z_3$ symmetry. In the present work we offer a model that contains the basic mechanisms behind this idea. The model will ultimately lead to the scenario of a domain wall hosting a network of defects, which may have direct interest to physics, as we show when we explore the pattern of the nested network in the case we allow the underlying $Z_2 \times Z_3$ symmetry to be broken.

We first develop the idea of a domain wall hosting a network of defects, in a model described by three real scalar fields, with the (dimensionless) potential,

$$V(\sigma, \phi, \chi) = \frac{2}{3} \left(\sigma^2 - \frac{9}{4} \right)^2 + \left(r \sigma^2 - \frac{9}{4} \right) (\phi^2 + \chi^2) + (\phi^2 + \chi^2)^2 - \phi(\phi^2 - 3\chi^2) \quad (1)$$

Here r couples σ to the pair of fields (ϕ, χ) . This potential is polynomial, and contains up to the fourth order power in the fields. Thus, it behaves standardly in $(3,1)$

space-time dimensions. Also, it presents discrete $Z_2 \times Z_3$ symmetry. We set $(\phi, \chi) \rightarrow (0, 0)$, to get the projection $V(\sigma, 0, 0) \rightarrow V(\sigma) = (2/3)(\sigma^2 - 9/4)^2$. The projected potential presents Z_2 symmetry, and can be written with the superpotential $W(\sigma) = (2\sqrt{3}/9)\sigma^3 - (3\sqrt{3}/2)\sigma$, in the form $V = (1/2)(dW/d\sigma)^2$. The reduced model supports the explicit configurations $\sigma_h(z) = \pm(3/2)\tanh(\sqrt{3}z)$. The tension of the host wall is $t_h = 3\sqrt{3} = (3/2)m_h$, where m_h represents the mass of the elementary σ meson. Also, the width of the wall is such that $l_h \sim 1/\sqrt{3}$.

The potentials projected inside ($\sigma \rightarrow 0$) and outside ($\sigma \rightarrow \pm 3/2$) the host domain wall are $V_{in}(\phi, \chi)$ and $V_{out}(\phi, \chi)$. Inside the wall we have

$$V_{in}(\phi, \chi) = (\phi^2 + \chi^2)^2 - \phi(\phi^2 - 3\chi^2) - \frac{9}{4}(\phi^2 + \chi^2) + \frac{27}{8} \quad (2)$$

This potential engenders the Z_3 symmetry, and there are three global minima, at the points $v_1^{in} = (3/2)(1, 0)$ and $v_{2,3}^{in} = (3/4)(-1, \pm\sqrt{3})$, which define an equilateral triangle. Outside the wall we get

$$V_{out}(\phi, \chi) = (\phi^2 + \chi^2)^2 - \phi(\phi^2 - 3\chi^2) + \frac{9}{4}(r-1)(\phi^2 + \chi^2) \quad (3)$$

V_{out} also engenders the Z_3 symmetry, but now the minima depend on r . We can adjust r such that $r > 9/8$, which is the condition for the fields ϕ and χ to develop no non-zero vacuum expectation value outside the host domain wall, ensuring that the model supports no domain defect outside the host domain wall. The restriction of considering quartic potentials forbids the possibility of describing the Z_3 portion of the model with the complex superpotential used in [21]; see also Ref. [22].

We investigate the masses of the elementary ϕ and χ mesons. Inside the wall they degenerate to the single value $m_{in} = 3\sqrt{3}/2$. Outside the wall, for $r > 9/8$ they also degenerate to a single value, $m_{out}(r) = 3\sqrt{(r-1)/2}$, which depends on r . We see that $m_{out}(r=4) = m_{in}$. Also, $m_{out}(r) > m_{in}$ for $r > 4$, and $m_{out}(r) < m_{in}$ for r in the interval $(9/8, 4)$.

We study linear stability of the classical solutions $\sigma = \sigma_h(z)$ and $(\phi, \chi) = (0, 0)$. The fields ϕ and χ vanish classically, and their fluctuations (η_n, ξ_n) decouple. The procedure leads to two equations for the fluctuations, that degenerate to the single Schrödinger-like equation

$$-\frac{d^2\psi_n(z)}{dz^2} + \frac{9}{2}V(z)\psi_n(z) = w_n^2\psi_n(z) \quad (4)$$

Here $V(z) = -1 + r \tanh^2 \sqrt{3}z$. This equation is of the modified Pöschl-Teller type, and can be examined analytically. The lowest eigenvalue is $w_0^2 = (3/2)\sqrt{6r+1} - 6$. There is instability for $r < 5/2$, showing that the host domain wall with $(\phi, \chi) = (0, 0)$ is unstable and therefore relax to lower energy configurations, with $(\phi, \chi) \neq (0, 0)$

for $r < 5/2$. Inside the host domain wall the sigma field vanishes, and the model is governed by the potential $V_{in}(\phi, \chi)$, which consequently may allow the presence of non-trivial (ϕ, χ) configurations. The host domain wall entraps the system described by $V_{in}(\phi, \chi)$ for the parameter r in the interval $(9/8, 5/2)$. In this interval we have $m_{out} < m_{in}$, showing that it is not energetically favorable for the elementary ϕ and χ mesons to live inside the wall for $r \in (9/8, 5/2)$. The model automatically suppress backreactions of the ϕ and χ mesons into the defects that may appear inside the host domain wall.

In Ref. [20] the potential inside the wall was shown to admit a network of domain walls, in the form of a hexagonal array of domain walls. In the thin wall approximation the network may be represented by the solutions

$$\phi_1 = \frac{3}{8} + \frac{9}{8} \tanh\left(\frac{1}{2}\sqrt{\frac{27}{8}}(y + \sqrt{3}x)\right) \quad (5)$$

$$\chi_1 = \frac{3}{8}\sqrt{3} - \frac{3}{8}\sqrt{3} \tanh\left(\frac{1}{2}\sqrt{\frac{27}{8}}(y + \sqrt{3}x)\right) \quad (6)$$

and by (ϕ_k, χ_k) , obtained by rotating the pair (ϕ_1, χ_1) by $2(k-1)\pi/3$, for $k = 2, 3$. We identify the space (ϕ, χ) with (x, y) , so rotations in (ϕ, χ) also rotates the plane (x, y) accordingly. The energy or tension of the individual defects in the network is given by, in the thin wall approximation $t_n = (27/8)\sqrt{3/2} = (9/8)m_{in}$. In the nested network, the width of each defect obeys $l_n \sim \sqrt{8/27}$. This shows that $l_h/l_n = 3/2\sqrt{2}$, and so the host domain wall is slightly thicker than the defects in the nested network. In the thin wall approximation, the potential $V_{in}(\phi, \chi)$ allows the formation of three-junctions as reactions that occur exothermically, and the nested array of thin wall configurations is stable. In Fig. 1 we depict the hexagonal network of defects inside the domain wall, in the thin wall approximation. The dashed lines show equilateral triangles, that belong to the dual lattice. Both the hexagonal network and the dual triangular network are composed of equilateral polygons, a fact that follows in accordance with the Z_3 symmetry.

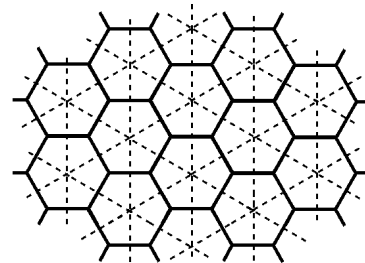


FIG. 1. The equilateral hexagonal network of defects, that may live inside the host domain wall. The dashed lines show the dual lattice, formed by equilateral triangles.

We now explore the breaking of the $Z_2 \times Z_3$ symmetry of the model. The simplest case refers to the breaking

of the Z_3 symmetry, without breaking the remaining Z_2 symmetry. We consider the case of breaking the internal Z_3 symmetry in the following way. We take for instance the vacuum state $v_1^{in} = (3/2)(1,0)$, and change its position to a location farther from or closer to the other minima of the system, increasing or decreasing the angle between two of the three defects. We illustrate this situation in Fig. 2. We do this without removing the degeneracy of the three minima. This mechanism changes $Z_3 \rightarrow Z_2$, so we refer to it as the minimal breaking.

We notice that the energy of the defect depends on the distance between the two minima the defect connects, and goes with the cube of it. If the Z_3 symmetry is broken to Z_2 , the three-junction is stable for $\cos(\theta/2) = t_n/2t'_n$, where θ is the angle between the modified defects, with tensions changed from t_n to t'_n . This implies that $t'_n > t_n$ for $\theta > 2\pi/3$, and $t'_n \leq t_n$ for $\theta \leq 2\pi/3$. If the vacuum state deviates significantly from its Z_3 -symmetric position, we cannot neglect the correction to the energy of the defects, and this would change the equilateral hexagons of Fig. 1 to non-equilateral hexagons. However, if the vacuum state deviates slightly from its Z_3 -symmetric position, one may neglect the correction to the energy of the defects. In this case we are slightly breaking $Z_3 \rightarrow Z_2$.

We now concentrate on breaking the Z_2 symmetry of the host domain wall. We can do this with the inclusion in the potential of a term odd in σ , that slightly removes the degeneracy of the two minima $\sigma = \pm 3/2$. Thus, the host domain wall bends trying to involve the local minimum, the false vacuum. To stabilize the non-topological structure we include charged fields into the system. The way one couples the charged fields is not unique, but if we choose to add fermions, we can couple them to the σ field in a way such that the projection with $(\phi, \chi) \rightarrow (0, 0)$ may leave the model supersymmetric. This is obtained with the superpotential $W(\sigma)$, with the Yukawa coupling $d^2W/d\sigma^2 = (4/3)\sqrt{3}\sigma$. In this case massless fermions bind [23] to the host domain wall, and contribute to stabilize [17] the non-topological defect that emerges with the breaking of the Z_2 symmetry.

The breaking of the Z_2 symmetry can be done breaking or not the remaining Z_3 symmetry of the model. We examine these two possibilities supposing that the host domain wall bends under the assumption of spherical symmetry, becoming a non-topological defect with the standard spherical shape. This is the minimal surface of genus zero, and according to the Euler theorem we can only tile the spherical surface with three-junctions as a regular polygonal network in the three different ways: with 4 triangles, or 6 squares, or yet 12 pentagons. These three cases preserve the Z_3 symmetry of the original network, locally, at the three-junction points. However, if locally one slightly breaks the Z_3 symmetry of the network to the Z_2 one, the three-junctions can now tile the spherical surface with 12 pentagons and 20 hexagons. We think of breaking the Z_3 symmetry minimally, to the Z_2 symmetry, through the same mechanism presented in Fig. 2.

Thus, if the symmetry is broken slightly we can consider the defect tensions as in the regular hexagonal network.

The tiling with 12 pentagons and 20 hexagons generates a spherical structure that resembles the fullerene, the buckyball composed of sixty carbon atoms. We visualize the symmetries involved in the spherical structures thinking of the corresponding dual lattices, which are triangular lattices, but in the three first cases the triangles are equilateral, while in the fourth case they are isosceles. We recall that regular heptagons introduce negative curvature, so they cannot appear when the genus zero surface is minimal. However, they may for instance spring to generate higher energy states from the fullerene-like structure, locally roughening the otherwise smooth spherical surface.

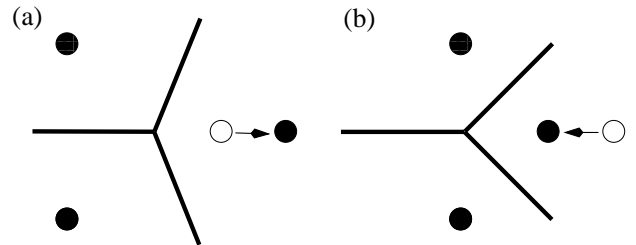


FIG. 2. The vacuum states (black dots) and the junction that forms the nested network. (a) and (b) illustrate the only two ways of breaking the $Z_3 \rightarrow Z_2$ symmetry.

We write the energy of the non-topological structure as $E_{nt}^n = E_{nt}^s + E_n$, where E_{nt}^s stands for the energy of the standard non-topological defect, and E_n is the portion due to the nested network. We use $E_{nt}^s = E_q + E_h$, which shows the contributions of the charged fields and of the host domain wall, respectively. We have $E_h = S t_h$ and $E_n = N d t_n$, where S is the area of the spherical surface, and N and d are the number and length of segments in the nested network. In the thin wall approximation the radius R of the non-topological structure should obey $R \gg l_h$, to make such structure much larger than the characteristic width of the host domain wall. We introduce the ratio

$$\frac{E_{nt}^n}{E_{nt}^s} = \left(1 + \frac{N}{1 + r_q}\right) \frac{t_n}{t_h} \frac{d}{S} \quad (7)$$

with $r_q = E_q/E_h$. The non-topological structure nests a network of defects, which modifies the scenario one gets with the standard domain wall. The modification depends on the way one couples charged bosons and fermions to the σ, ϕ , and χ fields. However, if the Z_3 symmetry is locally broken to the Z_2 one, the most probable defect corresponds to the fullerene or buckyball structure. But if the Z_3 symmetry is locally effective, there may be three equilateral structures, the most probable arising as follows. We consider the simpler case of plane polygonal structures, identifying the tetrahedron ($i = 3$), cube ($i = 4$), and dodecahedron ($i = 5$). We introduce R_{ij} as the energy ratio for the i and j structures. We get

$$R_{ij} = \frac{1 + r_q + \frac{9\sqrt{2}}{16h_i}}{1 + r_q + \frac{9\sqrt{2}}{16h_j}}, \quad i, j = 3, 4, 5 \quad (8)$$

Here h_3 , h_4 , and h_5 stand for the radius of the *incircle* of the triangle, square, and pentagon, respectively. Energy favors the tetrahedron, which is self-dual because the network and its dual are the very same triangular lattice. There are two other configurations, the octahedron, dual to the cube, and the icosahedron, dual to the dodecahedron. They do not appear in the $Z_2 \times Z_3$ model because they require four- and five-junctions, respectively.

The present work can be extended in several directions. For instance, in the $Z_2 \times Z_3$ model, if the host domain wall bends cylindrically, one may get to nanotube-like configurations [24]. Moreover, we could consider other models, presenting the $Z_2 \times Z_k$ symmetry ($k = 4, 5, 6$), leading to k -junctions. This would allow to tile the plane with squares ($k = 4$), or triangles ($k = 6$), and the spherical surface with triangles, as the octahedron ($k = 4$) or the icosahedron ($k = 5$). These investigations show a new way of modifying the standard domain wall, and this may find direct application in condensed matter [1–3] and in cosmology [4,5]. In cosmology, for instance, the internal network modifies the wall energy and changes its cosmological evolution. We follow Ref. [15] to see that the ratio between the volume pressure and the surface tension that govern the wall evolution now changes according to $(p_V/p_T)^n = (p_V/p_T)^s/(1 + \alpha t_n)$; $\alpha > 0$ is a numerical factor, and $t_n \rightarrow 0$ recovers the standard case. This shows that the ending scenario now depends not only on the way the symmetry is broken, but also on the modification the internal network introduces, enlarging the possibility of the wall dominating the energy density before the volume pressure can act. Another line could follow Ref. [14], asking how fermions could be coupled such that the corresponding zero modes that inhabit the wall could also bind to the internal network. Such mechanism would lead to a scenario where the nested fermions could form a one-dimensional gas inside the nested network, changing the way the soliton star evolves. We also mention the investigation concerning pattern formation in cosmology, as in the recent work [25]. Our investigation provides other possible scenarios for pattern formation in the early universe.

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